Forces Associated with Nonlinear Nonholonomic Constraint Equations

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Abstract

In previous work, a comprehensive, consistent, and concise method has been formulated for identifying a set of forces needed to constrain the behavior of a mechanical system modeled as a set of particles and rigid bodies. In this paper it is shown that the method is applicable to motion constraints described by nonholonomic equations that are inherently nonlinear in velocity. Two new approaches are presented for deriving equations governing motion of a system subject to such constraints. By using partial accelerations instead of the partial velocities normally employed with Kane's method, it is possible to form dynamical equations that either do or do not contain evidence of the constraint forces, depending on the analyst's interests.

1 Introduction

Motion constraints imposed on a mechanical system are described with nonholonomic (non-integrable) constraint equations, whereas configuration constraints are expressed with holonomic constraint equations. Two examples of motion constraints with which the reader may be familiar are the condition of rolling, which is the absence of slipping, and the restriction on velocity imposed by a sharp-edged blade. These constraints are sometimes described with equations written in the matrix form $\alpha u + \beta = 0$, where u is a column matrix of motion variables $u_1, \ldots, u_n$. Motion variables, also referred to as generalized speeds, are in general linear

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combinations of the time derivatives of generalized coordinates, $\dot{q}_1, \ldots, \dot{q}_n$. The distinguishing feature of such equations is that they are linear in the motion variables. Roberson and Schwertassek (Ref. [1]) note that all known motion constraints imposed on purely mechanical systems can be expressed with relationships that are linear in velocity variables. However, one may consider motion constraints that must be described by relationships that are inherently nonlinear in the motion variables, having the form $f(q_1, \ldots, q_n, u_1, \ldots, u_n, t) = 0$. In Ref. [2] Bajodah et al. review some of the literature dealing with nonlinear nonholonomic constraint equations and consider it important to study them because they can arise in connection with servo-constraints or program constraints when a control system enters the picture. As explained in Refs. [3] and [4], such constraints are enforced by application of control forces as opposed to the forces present when bodies and particles come into contact with one another, as is the case with classical, passive constraints.

Methods for dealing with nonlinear nonholonomic constraint equations are frequently illustrated by applying them to the Appell-Hamel mechanism. It is studied and discussed, for example, in Refs. [2] and [5] – [10]; however, it is known that the constraints imposed on this mechanical system can be expressed with linear relationships. In Refs. [11] and [12], Zekovich offers several examples of systems in which the constraints can be described with nonlinear nonholonomic constraint equations. Each example involves planar motion of two particles connected by a massless rigid rod or by a massless prismatic joint. Sharp-edged blades are attached in various ways so as to cause the velocities of the particles in an inertial reference frame N to be parallel, equal in magnitude, or perpendicular. In what follows it is shown that the associated constraints can in fact be expressed with linear nonholonomic equations. However, when the particles are not physically connected and the constraints are dictated by means other than the blades, the relationships expressing such restrictions on the velocities are inherently nonlinear.

The literature contains additional instances of nonlinear nonholonomic constraint equations. Another case of planar motion of two particles with parallel velocities, which serves as an example in Refs. [7], [13], [14], and [15], is brought about with a device proposed by Benenti in Ref. [16]. Benenti's mechanism consists of six rigid rods, one revolute joint, two blades, and a number of prismatic joints. Eight prismatic joints appear to be indicated in the figure in Ref. [16]; however, it is believed that there must be two successive prismatic joints in each of the locations indicated if all rods are to be able to move relative to each other. Jankowski provides two examples in Ref. [17] involving a single particle moving in a vertical plane subject to a uniform gravitational field and air resistance; the magnitude of the particle's velocity in N, or the magnitude of the acceleration in N, must match a prescribed time history. References [13], [8], [18], and [19] include an example proposed by Appell in which the velocity ${}^N\mathbf{v}^P$ in N of a particle P must satisfy the relationship $v_3{}^2 = a^2(v_1{}^2 + v_2{}^2)$, where a is a constant and v_r are the dot products of ${}^N\mathbf{v}^P$ with a set of right-handed, mutually perpendicular unit vectors $\hat{\mathbf{n}}_r$ fixed in N $(r = 1, 2, 3)$. Control of an inverted pendulum constitutes an example studied in Refs. [13] and [14]. A thin rigid rod moves in a vertical plane in the presence of a uniform gravitational field, with the lower end of the rod always in contact with a horizontal line. The system is referred to as Marle's servomechanism; as proposed in Ref. [8], an actuator controls the horizontal displacement of the rod's lower end according to some control law in order to keep the rod vertical. An earlier paper by Huston and Passerello (Ref. [20]) considers the more general case of balancing a pole whose lower end remains in contact with a horizontal plane, while the pole is otherwise free to move in the space above the horizontal plane.

In Ref. [21], a comprehensive, consistent, and concise method is established for identifying a set of forces needed to constrain the behavior of a mechanical system modeled as a set of particles and rigid bodies. The purpose of this paper is to apply the method to constraints described by nonholonomic equations that are inherently nonlinear in velocity. (It is to be understood that the term "velocity," used in the general case of a system of particles, subsumes "angular velocity" in the special case in which a subset of particles makes up a rigid body. The term "acceleration" likewise encompasses an angular counterpart.) An essential feature of the method consists of expressing constraint equations in vector form rather than entirely in terms of scalars. A constraint equation that has been differentiated once or twice with respect to time, so that it contains the acceleration of a point or the angular acceleration of a rigid body, is said to be written at the acceleration level. Likewise, a constraint equation at the velocity level is one that has been differentiated at most once, so that it contains the velocity of a point or the angular velocity of a rigid body. The method developed in Ref. [21] is applied in that work to configuration constraints, and to motion constraints that are linear in velocity when expressed at the velocity level. It so happens that the method can be applied whenever constraints can be described at the acceleration level by a set of independent equations that are linear in acceleration; therefore, it is applicable to constraint equations that are nonlinear in velocity when written at the velocity level.

The remainder of the paper is organized as follows. First, a treatment of nonlinear nonholonomic constraint equations is undertaken in Sec. 2 for a generic system of particles; the results are applicable whether or not a subset of particles makes up a rigid body. The method of Ref. [21] is used to identify directions of constraint forces and the particles to which they must be applied. The constraint forces are used together with extensions to Kane's method (Ref. [22]) to obtain two new ways of deriving dynamical equations of motion. The first of these is useful when one is interested in the time histories of the constraint forces; it produces dynamical equations that contain evidence of the constraint forces needed to satisfy the nonlinear nonholonomic constraint equations. On the other hand, the second approach can be used when one is not interested in the constraint forces but requires dynamical equations governing the motion of the constrained system; constraint forces are not in evidence in the minimal equations of motion obtained with this approach. The novelty in each case rests in the use of partial accelerations rather than the partial velocities employed in Kane's method. Both formulations are applied in Sec. 3 to an example in which the velocities of two particles must remain perpendicular. The resulting equations of motion are solved numerically. Constraint forces are identified in Sec. 4 for two other examples in which the velocities of two particles must either remain parallel, or equal in magnitude. A second demonstration of the two approaches for obtaining equations of motion is performed with Appell's particle in Sec. 5, and the equations are compared to existing results. Finally, the two approaches are adapted in Sec. 6 to the special case in which a system of particles contains a rigid body. Concluding remarks are supplied in Sec. 7.

2 Equations of Motion for Complex Nonholonomic Systems

It is instructive to recall that configuration constraints are, in general, expressed at the position level with nonlinear holonomic constraint equations. However, when these relationships are expressed at the velocity level they are linear in the velocity vectors or, what is the same, linear in the motion variables as shown in Ref. [21]. Similarly, motion constraints in general are described at the velocity level by nonlinear nonholonomic constraint equations but, when expressed at the acceleration level, they are linear in the acceleration vectors. In other words, when written in scalar form the latter relationships are linear in the time derivatives of motion variables. Two important conclusions follow from these observations. First, forces needed to satisfy nonlinear nonholonomic constraint equations can be formed with the approach described in Ref. [21]. Second, partial accelerations can be used in place of partial velocities to eliminate the constraint forces from equations of motion in which they appear.

Suppose that a simple nonholonomic system S (Ref. [22]) is made up of particles $P_1, \ldots, P_\nu$. The configuration of S in a Newtonian reference frame N is described by generalized coordinates $q_1, \ldots, q_n$, and the motion of S is characterized by independent motion variables $u_1, \ldots, u_p$. Suppose further that S is subject to ℓ nonlinear nonholonomic constraint equations

$$h_s({}^N\mathbf{v}^{P_1}, \ldots, {}^N\mathbf{v}^{P_\nu}, t) = 0 \qquad (s = 1, \ldots, \ell) \tag{1}$$

where ${}^N\mathbf{v}^{P_i}$ is the velocity of particle P_i ($i = 1, \ldots, \nu$) in a Newtonian reference frame N, and where t denotes time. In this case S is referred to as a *complex nonholonomic system*. Differentiation of these relationships with respect to t in N yields

$$\sum_{i=1}^{\nu} {}^N\mathbf{a}^{P_i} \cdot \mathbf{W}_{is} + Z_s = 0 \qquad (s = 1, \ldots, \ell) \tag{2}$$

where $\mathbf{W}_{is}$ are vector functions of $q_1, \ldots, q_n, u_1, \ldots, u_p$ and t in N, and Z_s are scalar functions of the same variables. The acceleration of P_i in N is represented by ${}^N\mathbf{a}^{P_i}$. When these relationships are satisfied the motion variables $u_1, \ldots, u_p$ are no longer independent, as discussed presently. According to Ref. [21] one can inspect these relationships and conclude that constraint forces are given by

$$\mathbf{C}_{is} = \lambda_s \mathbf{W}_{is} \qquad (i = 1, \ldots, \nu;\ s = 1, \ldots, \ell) \tag{3}$$

where λ_s are scalar multipliers whose time histories may be of interest. As discussed in Ref. [21], $\mathbf{C}_{is}$ is parallel to $\mathbf{W}_{is}$ and in general it must be applied to P_i in order to satisfy the constraint equations (2).

Dynamical equations of motion to which $\mathbf{C}_{is}$ do contribute are given by

$$\begin{aligned} \tilde{F}_r + \tilde{F}_r^\star &= \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \left(\mathbf{R}_i - m_i {}^N\mathbf{a}^{P_i}\right) \\ &= \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \left(\mathbf{f}_i + \sum_{s=1}^{\ell} \lambda_s \mathbf{W}_{is} - m_i {}^N\mathbf{a}^{P_i}\right) = 0 \qquad (r = 1, \ldots, p) \end{aligned} \tag{4}$$

where $\tilde{F}_r$, $\tilde{F}_r^\star$, and ${}^N\tilde{\mathbf{v}}_r^{P_i}$ respectively denote the rth nonholonomic generalized active force for S in N, nonholonomic generalized inertia force for S in N, and nonholonomic partial velocity of P_i in N (Ref. [22]). The mass of P_i is indicated by m_i. The resultant $\mathbf{R}_i$ of all contact forces and distance forces acting on P_i is regarded as the sum of the constraint forces $\sum_{s=1}^{\ell} \lambda_s \mathbf{W}_{is}$ that must be applied to ensure satisfaction of Eqs. (2), added to the resultant of all other forces, $\mathbf{f}_i$. Equations (4) together with Eqs. (2) furnish the number of relationships needed to solve for the unknown quantities $\dot{u}_1, \ldots, \dot{u}_p, \lambda_1, \ldots, \lambda_\ell$. One employs these relationships if the time histories of $\lambda_1, \ldots, \lambda_\ell$ are of interest.

A reduced or minimal set of dynamical equations to which $\mathbf{C}_{is}$ do not contribute is given by

$$\begin{aligned} \tilde{\tilde{F}}_r + \tilde{\tilde{F}}_r^\star &= \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \left(\mathbf{f}_i + \sum_{s=1}^{\ell} \lambda_s \mathbf{W}_{is} - m_i {}^N\mathbf{a}^{P_i}\right) \\ &= \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \left(\mathbf{f}_i - m_i {}^N\mathbf{a}^{P_i}\right) = 0 \qquad (r = 1, \ldots, c) \end{aligned} \tag{5}$$

where

$$c \triangleq p - \ell \tag{6}$$

is the number of degrees of freedom of S in N. When speaking of $\tilde{\tilde{F}}_r$ and $\tilde{\tilde{F}}_r^\star$ it is convenient to refer to them, respectively, as the rth nonholonomic generalized active force and the

rth nonholonomic generalized inertia force, but the double tilde notation should be used to indicate they have been formed with ${}^N\tilde{\mathbf{a}}_r^{P_i}$, the rth *nonholonomic partial acceleration* of P_i in N, rather than ${}^N\tilde{\mathbf{v}}_r^{P_i}$. Instructions for obtaining nonholonomic partial accelerations are now given, and their role in eliminating the multipliers from Eqs. (5) is discussed.

The acceleration of P_i in N can be written uniquely as

$$ {}^N\mathbf{a}^{P_i} = \sum_{r=1}^{p} {}^N\mathbf{a}_r^{P_i}\,\dot{u}_r + {}^N\mathbf{a}_t^{P_i} \qquad (i = 1, \ldots, \nu) \tag{7} $$

and also uniquely as

$$ {}^N\mathbf{a}^{P_i} = \sum_{r=1}^{c} {}^N\tilde{\mathbf{a}}_r^{P_i}\,\dot{u}_r + {}^N\tilde{\mathbf{a}}_t^{P_i} \qquad (i = 1, \ldots, \nu) \tag{8} $$

The first of these expressions can be obtained from Eqs. (2.14.4) of Ref. [22] by differentiation with respect to t in N, in which case the partial acceleration ${}^N\mathbf{a}_r^{P_i}$ is seen to be identical to the nonholonomic partial velocity of P_i in N,

$$ {}^N\mathbf{a}_r^{P_i} \triangleq {}^N\tilde{\mathbf{v}}_r^{P_i} \qquad (i = 1, \ldots, \nu;\ r = 1, \ldots, p) \tag{9} $$

and the acceleration remainder ${}^N\mathbf{a}_t^{P_i}$ is defined to be

$$ {}^N\mathbf{a}_t^{P_i} \triangleq \sum_{r=1}^{p} \left(\frac{{}^N d}{dt}\, {}^N\tilde{\mathbf{v}}_r^{P_i} \right) u_r + \frac{{}^N d}{dt}\, {}^N\tilde{\mathbf{v}}_t^{P_i} \qquad (i = 1, \ldots, \nu) \tag{10} $$

Substitution from Eqs. (7) into (2) gives

$$ \sum_{r=1}^{p} \left(\sum_{i=1}^{\nu} {}^N\mathbf{a}_r^{P_i} \cdot \mathbf{W}_{is} \right) \dot{u}_r + \sum_{i=1}^{\nu} {}^N\mathbf{a}_t^{P_i} \cdot \mathbf{W}_{is} + Z_s = 0 \qquad (s = 1, \ldots, \ell) \tag{11} $$

The coefficients of $\dot{u}_r$ and the remaining terms can be abbreviated respectively by means of two definitions,

$$ \alpha_{sr} \triangleq \sum_{i=1}^{\nu} {}^N\mathbf{a}_r^{P_i} \cdot \mathbf{W}_{is} \qquad (s = 1, \ldots, \ell;\ r = 1, \ldots, p) \tag{12} $$

and

$$ \gamma_s \triangleq Z_s + \sum_{i=1}^{\nu} {}^N\mathbf{a}_t^{P_i} \cdot \mathbf{W}_{is} \qquad (s = 1, \ldots, \ell) \tag{13} $$

where α_{sr} and γ_s are functions of $q_1, \ldots, q_n, u_1, \ldots, u_p$, and the time t. These definitions allow Eqs. (11) to be rewritten in a form that is linear in the time derivatives of the motion variables

$$ \sum_{r=1}^{p} \alpha_{sr}\dot{u}_r + \gamma_s = 0 \qquad (s = 1, \ldots, \ell) \tag{14} $$

These relationships express the dependence of ℓ time derivatives of the motion variables, say $\dot{u}_{c+1}, \ldots, \dot{u}_p$, on the remaining ones $\dot{u}_1, \ldots, \dot{u}_c$. It is assumed that this partitioning is such that these equations can in fact be solved for $\dot{u}_{c+1}, \ldots, \dot{u}_p$ in terms of $\dot{u}_1, \ldots, \dot{u}_c$. With a relationship for ${}^N\mathbf{a}^{P_i}$ in hand having the form of Eqs. (7), one simply embeds the acceleration level constraint equations by rewriting $\dot{u}_{c+1}, \ldots, \dot{u}_p$ in terms of $\dot{u}_1, \ldots, \dot{u}_c$ to obtain an expression in the form of Eqs. (8). Nonholonomic partial accelerations ${}^N\tilde{\mathbf{a}}_r^{P_i}$ are subsequently obtained in the same way as partial velocities, namely by inspecting the resulting relationship for acceleration to determine the vector coefficients of $\dot{u}_r$ for $r = 1, \ldots, c$.

The dependent motion variable time derivatives are written in terms of the independent ones in a manner analogous to Eqs. (2.13.1) of Ref. [22],

$$\dot{u}_{c+r} = \sum_{s=1}^{c} A_{rs}\dot{u}_s + B_r \qquad (r = 1, \ldots, \ell) \tag{15}$$

To bring Eqs. (14) into this form, begin by putting Eqs. (15) in matrix form,

$$\dot{u}_D = A\dot{u}_I + B \tag{16}$$

where $\dot{u}_I$ is a $c \times 1$ column array containing the independent quantities $\dot{u}_1, \ldots, \dot{u}_c$, $\dot{u}_D$ is an $\ell \times 1$ column array containing the dependent quantities $\dot{u}_{c+1}, \ldots, \dot{u}_p$, A is an $\ell \times c$ matrix whose elements are A_{rs}, and B is an $\ell \times 1$ column array with elements B_r. Borrowing from the strategy of generalized coordinate partitioning (Refs. [23] and [24]), Eqs. (14) can be recast in matrix form with motion variable time derivative partitioning as

$$\alpha_I\dot{u}_I + \alpha_D\dot{u}_D + \gamma = 0 \tag{17}$$

where α_I is an $\ell \times c$ matrix, α_D is an $\ell \times \ell$ matrix, and γ is an $\ell \times 1$ column array whose elements are $\gamma_1, \ldots, \gamma_\ell$. The motion variable time derivatives can always be ordered such that α_D has an inverse as long as the constraint equations are independent, thus

$$\dot{u}_D = -{\alpha_D}^{-1}\alpha_I\dot{u}_I - {\alpha_D}^{-1}\gamma \tag{18}$$

and comparison of this relationship with Eq. (16) produces the definitions

$$A \triangleq -{\alpha_D}^{-1}\alpha_I, \qquad B \triangleq -{\alpha_D}^{-1}\gamma \tag{19}$$

We are now in a position to undertake two instructive exercises. The first is to determine the contribution of the constraint forces $\mathbf{C}_{is}$ $(i = 1, \ldots, \nu; s = 1, \ldots, \ell)$ to the nonholonomic generalized active forces $\tilde{F}_r$ $(r = 1, \ldots, p)$. The second is to show, in general, that $\mathbf{C}_{is}$ contribute nothing to the nonholonomic generalized active forces $\tilde{\tilde{F}}_r$ $(r = 1, \ldots, c)$.

Nonholonomic generalized active forces for S in N are defined by Eqs. (4.4.1) in Ref. [22]:

$$\tilde{F}_r \triangleq \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{R}_i \qquad (r = 1, \ldots, p) \tag{20}$$

Let $\mathbf{C}_i$ represent the resultant of the constraint forces $\mathbf{C}_{is}$ applied to P_i in order to ensure satisfaction of Eqs. (2), so that

$$\mathbf{C}_i \triangleq \sum_{s=1}^{\ell} \mathbf{C}_{is} = \sum_{s=1}^{\ell} \lambda_s \mathbf{W}_{is} \qquad (i = 1, \ldots, \nu) \tag{21}$$

The resultant $\mathbf{R}_i$ of all contact forces and distance forces acting on P_i can then be regarded as the sum of the constraint force $\mathbf{C}_i$ and the resultant of all other forces, $\mathbf{f}_i$. Hence, $\tilde{F}_r$ is made up of contributions $(\tilde{F}_r)_{\mathcal{C}}$ from the constraint forces acting on S and $(\tilde{F}_r)_{\mathcal{F}}$ from all other forces acting on S,

$$\tilde{F}_r = (\tilde{F}_r)_{\mathcal{C}} + (\tilde{F}_r)_{\mathcal{F}} \triangleq \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{C}_i + \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{f}_i \qquad (r = 1, \ldots, p) \tag{22}$$

The contribution from the constraint forces can be singled out, and it is given by

$$(\tilde{F}_r)_{\mathcal{C}} = \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \sum_{s=1}^{\ell} \lambda_s \mathbf{W}_{is} = \sum_{s=1}^{\ell} \lambda_s \alpha_{sr} \qquad (r = 1, \ldots, p) \tag{23}$$

where α_{sr} has the same meaning as in Eqs. (12).

It can now be shown that the constraint forces $\mathbf{C}_{is}$ make no contribution to any of $\tilde{\tilde{F}}_r$. Equations (23) can be expressed in matrix form as

$$(\tilde{F})_{\mathcal{C}} = \alpha^{\mathrm{T}} \lambda = \begin{Bmatrix} {\alpha_I}^{\mathrm{T}} \lambda \\ {\alpha_D}^{\mathrm{T}} \lambda \end{Bmatrix} \triangleq \begin{Bmatrix} (\tilde{F}_I)_{\mathcal{C}} \\ (\tilde{F}_D)_{\mathcal{C}} \end{Bmatrix} \tag{24}$$

where λ is an $\ell \times 1$ column array whose elements are $\lambda_1, \ldots, \lambda_\ell$, $(\tilde{F}_I)_{\mathcal{C}}$ is a $c \times 1$ column array with elements $(\tilde{F}_1)_{\mathcal{C}}, \ldots, (\tilde{F}_c)_{\mathcal{C}}$, and $(\tilde{F}_D)_{\mathcal{C}}$ is an $\ell \times 1$ column array with elements $(\tilde{F}_{c+1})_{\mathcal{C}}, \ldots, (\tilde{F}_p)_{\mathcal{C}}$. In view of the analogous relationship between Eqs. (2.13.1) of Ref. [22] and Eqs. (15), one can write a relationship analogous to Eqs. (4.4.3) in Ref. [22],

$$(\tilde{\tilde{F}}_r)_{\mathcal{C}} = (\tilde{F}_r)_{\mathcal{C}} + \sum_{s=1}^{\ell} (\tilde{F}_{c+s})_{\mathcal{C}} \; A_{sr} \qquad (r = 1, \ldots, c) \tag{25}$$

These relationships can be expressed in matrix form as

$$
\begin{aligned}
(\tilde{\tilde{F}})_{\mathcal{C}} &= (\tilde{F}_I)_{\mathcal{C}} + A^{\mathrm{T}}(\tilde{F}_D)_{\mathcal{C}} \\
&= {\alpha_I}^{\mathrm{T}}\lambda + A^{\mathrm{T}}{\alpha_D}^{\mathrm{T}}\lambda = \left({\alpha_I}^{\mathrm{T}} + A^{\mathrm{T}}{\alpha_D}^{\mathrm{T}}\right)\lambda
\end{aligned}
\tag{26}
$$

The term in parentheses is observed to vanish by noting

$$
0 = \alpha_I - \alpha_I = \alpha_I - \alpha_D{\alpha_D}^{-1}\alpha_I = \alpha_I + \alpha_D A \tag{27}
$$

Hence, the transpose of this relationship is ${\alpha_I}^{\mathrm{T}} + A^{\mathrm{T}}{\alpha_D}^{\mathrm{T}} = 0$. This step may be viewed as premultiplication of α^{T} by an orthogonal complement matrix $[I_c \ A^{\mathrm{T}}]$, where I_c is the $c \times c$ identity matrix. In any event, it is shown that

$$
(\tilde{\tilde{F}}_r)_{\mathcal{C}} = 0 \qquad (r = 1, \ldots, c) \tag{28}
$$

or, in words, motion constraints described by inherently nonlinear nonholonomic constraint equations require the application of forces that make no contributions to any of the nonholonomic generalized active forces $\tilde{\tilde{F}}_r$. Because these contributions are defined in terms of nonholonomic partial accelerations ${}^N\tilde{\mathbf{a}}_r^{P_i}$ as

$$
(\tilde{\tilde{F}}_r)_{\mathcal{C}} = \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \mathbf{C}_i = \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \sum_{s=1}^{\ell} \lambda_s \mathbf{W}_{is} = \sum_{s=1}^{\ell} \lambda_s \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \mathbf{W}_{is} \qquad (r = 1, \ldots, c) \tag{29}
$$

it can be concluded that

$$
\sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \mathbf{W}_{is} = 0 \qquad (r = 1, \ldots, c;\ s = 1, \ldots, \ell) \tag{30}
$$

3 Two Particles with Perpendicular Velocities

An example is provided to illustrate application of Eqs. (4) and (5) to form equations of motion in which constraint forces respectively are and are not in evidence. A system of two individual particles is subject to a requirement that the velocity in a Newtonian reference frame N of one particle must remain perpendicular to the velocity in N of the other particle. The associated nonholonomic constraint equation is inherently nonlinear. Implementation of the constraint would require the sort of computations that are associated with a control system, as well as ideal actuators and sensors; thus, the example features a servo-constraint. The demonstration is followed by discussion of a similar example from the literature in which the constraint is imposed by purely mechanical means, and it is shown that the nonholonomic constraint equation can in that case be expressed as a linear relationship.

Two pucks moving on an air-bearing table fixed in a Newtonian reference frame N are modeled as particles P_1 with a mass of m_1, and P_2 with a mass of m_2. Let two orthogonal unit vectors $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ be fixed in N and define the plane of the table, and let unit vector $\hat{\mathbf{n}}_3 \triangleq \hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2$ be normal to the plane. An external force $\mathbf{f}_1 = \sigma_1 \hat{\mathbf{n}}_1 + \sigma_2 \hat{\mathbf{n}}_2$ is applied to P_1 whereas a force $\mathbf{f}_2 = \sigma_3 \hat{\mathbf{n}}_1 + \sigma_4 \hat{\mathbf{n}}_2$ is applied to P_2. The motion of this system is regarded as unconstrained. Suppose that the velocities ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ of P_1 and P_2 in N are to be constrained such that they must remain perpendicular at all times. Let $m_1 = 1$ kg, $m_2 = 2$ kg, and let $\mathbf{f}_1$ and $\mathbf{f}_2$ be characterized by the constants $\sigma_1 = 1.0$ N, $\sigma_2 = 0$ N, $\sigma_3 = 1.0$ N, and $\sigma_4 = 0$ N. At $t = 0$ the velocities of P_1 and P_2 in N are given by ${}^N\mathbf{v}^{P_1} = 0.3\hat{\mathbf{n}}_1 + 0.4\hat{\mathbf{n}}_2$ m/s, and ${}^N\mathbf{v}^{P_2} = 0.4\hat{\mathbf{n}}_1 - 0.3\hat{\mathbf{n}}_2$ m/s. The initial position vectors $\mathbf{p}_i$ from a point O fixed in N to P_i are given by $\mathbf{p}_1 = 1\hat{\mathbf{n}}_1 - 2\hat{\mathbf{n}}_2$ m, and $\mathbf{p}_2 = 1\hat{\mathbf{n}}_1 + 2\hat{\mathbf{n}}_2$ m.

First, a constraint equation is written at the acceleration level in vector form. It is inspected to construct expressions for the constraint forces that must be applied to P_1 and P_2 in order for the constraint to be obeyed. A constraint force can be applied to a puck, for example, by four orthogonally mounted thrusters. Equations (4) are then employed to produce dynamical equations of motion in which the constraint forces play a part, and these equations are solved numerically together with kinematical differential equations.

The constraint can be expressed by the relationship

$$
{}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_1} = 0 \tag{31}
$$

This constraint equation is nonlinear in the velocity vectors because more than one velocity appears in a dot product; it is also nonlinear in motion variables, as will become apparent. Differentiation with respect to t in N brings the constraint equation to the acceleration level, where it is seen to be linear in the acceleration vectors because only one such vector appears in each dot product.

$$ {}^N\mathbf{a}^{P_2} \cdot {}^N\mathbf{v}^{P_1} + {}^N\mathbf{a}^{P_1} \cdot {}^N\mathbf{v}^{P_2} = 0 \tag{32} $$

With Eqs. (2) and (3) in mind, it can be concluded that the constraint requires application of the forces

$$ \mathbf{C}_2 = \lambda\, {}^N\mathbf{v}^{P_1}, \qquad \mathbf{C}_1 = \lambda\, {}^N\mathbf{v}^{P_2} \tag{33} $$

to P_2 and P_1 respectively. The constraint forces $\mathbf{C}_1$ and $\mathbf{C}_2$ need not be of equal magnitudes because the constraint does not require ${}^N\mathbf{v}^{P_2}$ and ${}^N\mathbf{v}^{P_1}$ to be equal in magnitude. The constraint force $\mathbf{C}_1$ is perpendicular to $\mathbf{C}_2$ when the constraint is satisfied.

The unconstrained system possesses four degrees of freedom in N, thus the motion can be characterized by four motion variables defined operationally as

$$ {}^N\mathbf{v}^{P_1} = u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2, \qquad {}^N\mathbf{v}^{P_2} = u_3\hat{\mathbf{n}}_1 + u_4\hat{\mathbf{n}}_2 \tag{34} $$

Dynamical equations of motion formed according to Eqs. (4) are readily written as

$$ m_1\dot{u}_1 = \sigma_1 + \lambda u_3, \quad m_1\dot{u}_2 = \sigma_2 + \lambda u_4, \quad m_2\dot{u}_3 = \sigma_3 + \lambda u_1, \quad m_2\dot{u}_4 = \sigma_4 + \lambda u_2 \tag{35} $$

The constraint expressed at the velocity level in vector form by Eq. (31) becomes, in scalar form,

$$ u_1u_3 + u_2u_4 = 0 \tag{36} $$

This relationship is nonlinear in the motion variables. The constraint at the acceleration level is, however, linear in the time derivatives of the motion variables,

$$ u_3\dot{u}_1 + u_4\dot{u}_2 + u_1\dot{u}_3 + u_2\dot{u}_4 = 0 \tag{37} $$

An analytical solution of the linear system of equations (35) and (37) for the five unknowns is manageable, and is given by

$$ \lambda = -\frac{m_1(\sigma_3u_1 + \sigma_4u_2) + m_2(\sigma_1u_3 + \sigma_2u_4)}{m_1({u_1}^2 + {u_2}^2) + m_2({u_3}^2 + {u_4}^2)} \tag{38} $$

$$ \dot{u}_1 = \frac{\sigma_1 + \lambda u_3}{m_1}, \quad \dot{u}_2 = \frac{\sigma_2 + \lambda u_4}{m_1}, \quad \dot{u}_3 = \frac{\sigma_3 + \lambda u_1}{m_2}, \quad \dot{u}_4 = \frac{\sigma_4 + \lambda u_2}{m_2} \tag{39} $$

The configuration of P_1 and P_2 in N is described by four generalized coordinates introduced operationally as

$$\mathbf{p}_1 = q_1\hat{\mathbf{n}}_1 + q_2\hat{\mathbf{n}}_2, \qquad \mathbf{p}_2 = q_3\hat{\mathbf{n}}_1 + q_4\hat{\mathbf{n}}_2 \tag{40}$$

Four kinematical differential equations are given simply by

$$\dot{q}_r = u_r \qquad (r = 1, 2, 3, 4) \tag{41}$$

The unconstrained trajectories ($\lambda = 0$) of P_1 and P_2 are displayed in the upper left of Fig. 1, to be compared to the constrained trajectories shown in the upper right. It is clear that ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ are becoming parallel in the absence of constraint forces, whereas they remain perpendicular when $\mathbf{C}_1$ and $\mathbf{C}_2$ are applied. A time history of λ is shown in the lower left of Fig. 1. The constraint requires ${}^N\mathbf{v}^{P_2}$ to remain perpendicular to ${}^N\mathbf{v}^{P_1}$; hence, the cosine of the angle between the two vectors calculated as $\cos\theta = {}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_1}/(|{}^N\mathbf{v}^{P_2}||{}^N\mathbf{v}^{P_1}|)$, which should be 0, can be used as a measure of the failure of the numerical solution to satisfy the constraint. As seen in the lower right of Fig. 1, the solution meets the constraint very well.

One can virtually eliminate the small error evident in the time history of $\cos\theta$ and remove λ from the dynamical equations of motion by appealing to Eqs. (5). Starting with the accelerations in N of P_1 and P_2 expressed as

$${}^N\mathbf{a}^{P_1} = \dot{u}_1\hat{\mathbf{n}}_1 + \dot{u}_2\hat{\mathbf{n}}_2, \qquad {}^N\mathbf{a}^{P_2} = \dot{u}_3\hat{\mathbf{n}}_1 + \dot{u}_4\hat{\mathbf{n}}_2 \tag{42}$$

and substituting the expression for $\dot{u}_4$ obtained from Eq. (37), one arrives at

$${}^N\mathbf{a}^{P_1} = \dot{u}_1\hat{\mathbf{n}}_1 + \dot{u}_2\hat{\mathbf{n}}_2, \qquad {}^N\mathbf{a}^{P_2} = \dot{u}_3\hat{\mathbf{n}}_1 - \frac{1}{u_2}(u_3\dot{u}_1 + u_4\dot{u}_2 + u_1\dot{u}_3)\hat{\mathbf{n}}_2 \tag{43}$$

The nonholonomic partial accelerations of P_1 and P_2 in N are identified as

$${}^N\tilde{\mathbf{a}}_1^{P_1} = \hat{\mathbf{n}}_1, \qquad {}^N\tilde{\mathbf{a}}_2^{P_1} = \hat{\mathbf{n}}_2, \qquad {}^N\tilde{\mathbf{a}}_3^{P_1} = \mathbf{0} \tag{44}$$

$${}^N\tilde{\mathbf{a}}_1^{P_2} = -\frac{u_3}{u_2}\hat{\mathbf{n}}_2, \qquad {}^N\tilde{\mathbf{a}}_2^{P_2} = -\frac{u_4}{u_2}\hat{\mathbf{n}}_2, \qquad {}^N\tilde{\mathbf{a}}_3^{P_2} = \hat{\mathbf{n}}_1 - \frac{u_1}{u_2}\hat{\mathbf{n}}_2 \tag{45}$$

With these partial accelerations in hand, nonholonomic generalized active forces are formed according to the expressions

$$\overset{\approx}{F}_r = {}^N\tilde{\mathbf{a}}_r^{P_1} \cdot (\mathbf{f}_1 + \lambda\, {}^N\mathbf{v}^{P_2}) + {}^N\tilde{\mathbf{a}}_r^{P_2} \cdot (\mathbf{f}_2 + \lambda\, {}^N\mathbf{v}^{P_1}) \qquad (r = 1, 2, 3) \tag{46}$$

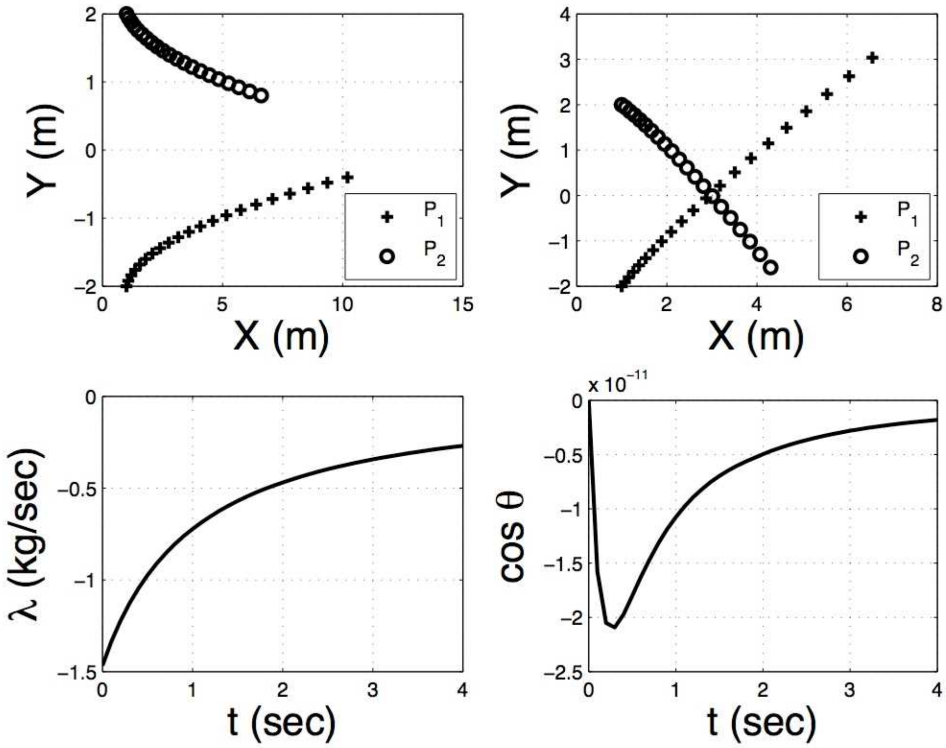

Figure 1: Two Particles with Perpendicular Velocities

The first of these is given by

$$\begin{aligned}\tilde{\tilde{F}}_1 &= \hat{\mathbf{n}}_1 \cdot (\mathbf{f}_1 + \lambda\, {}^N\mathbf{v}^{P_2}) - \frac{u_3}{u_2}\hat{\mathbf{n}}_2 \cdot (\mathbf{f}_2 + \lambda\, {}^N\mathbf{v}^{P_1}) \\ &= \sigma_1 + \lambda u_3 - \frac{u_3}{u_2}(\sigma_4 + \lambda u_2) \\ &= \sigma_1 - \frac{u_3}{u_2}\sigma_4 \end{aligned} \tag{47}$$

Similarly,

$$\tilde{\tilde{F}}_2 = \sigma_2 - \frac{u_4}{u_2}\sigma_4 \tag{48}$$

$$\tilde{\tilde{F}}_3 = \sigma_3 - \frac{u_1}{u_2}\sigma_4 \tag{49}$$

The multiplier λ is clearly eliminated from $\tilde{\tilde{F}}_1$, $\tilde{\tilde{F}}_2$, and $\tilde{\tilde{F}}_3$, and thus the constraint forces $\mathbf{C}_1$ and $\mathbf{C}_2$ do not contribute to the reduced equations of motion. Nonholonomic generalized inertia forces are given by

$$\tilde{\tilde{F}}_r^\star = {}^N\tilde{\mathbf{a}}_r^{P_1} \cdot (-m_1\, {}^N\mathbf{a}^{P_1}) + {}^N\tilde{\mathbf{a}}_r^{P_2} \cdot (-m_2\, {}^N\mathbf{a}^{P_2}) \qquad (r = 1, 2, 3) \tag{50}$$

or

$$\begin{aligned}\tilde{\tilde{F}}_1^\star &= -m_1\dot{u}_1 - m_2\frac{u_3}{{u_2}^2}(u_3\dot{u}_1 + u_4\dot{u}_2 + u_1\dot{u}_3) \\ &= -\left[m_1 + m_2\left(\frac{u_3}{u_2}\right)^2\right]\dot{u}_1 - m_2\frac{u_3u_4}{{u_2}^2}\dot{u}_2 - m_2\frac{u_1u_3}{{u_2}^2}\dot{u}_3\end{aligned} \tag{51}$$

$$\begin{aligned}\tilde{\tilde{F}}_2^\star &= -m_1\dot{u}_2 - m_2\frac{u_4}{{u_2}^2}(u_3\dot{u}_1 + u_4\dot{u}_2 + u_1\dot{u}_3) \\ &= -m_2\frac{u_3u_4}{{u_2}^2}\dot{u}_1 - \left[m_1 + m_2\left(\frac{u_4}{u_2}\right)^2\right]\dot{u}_2 - m_2\frac{u_1u_4}{{u_2}^2}\dot{u}_3\end{aligned} \tag{52}$$

$$\begin{aligned}\tilde{\tilde{F}}_3^\star &= -m_2\dot{u}_3 - m_2\frac{u_1}{{u_2}^2}(u_3\dot{u}_1 + u_4\dot{u}_2 + u_1\dot{u}_3) \\ &= -m_2\frac{u_1u_3}{{u_2}^2}\dot{u}_1 - m_2\frac{u_1u_4}{{u_2}^2}\dot{u}_2 - m_2\left[1 + \left(\frac{u_1}{u_2}\right)^2\right]\dot{u}_3\end{aligned} \tag{53}$$

The mass matrix associated with these equations of motion is symmetric. After expressing u_4 as $-u_1u_3/u_2$ as required by Eq. (36), the reduced dynamical equations of motion $\tilde{\tilde{F}}_r + \tilde{\tilde{F}}_r^\star = 0$ $(r = 1, 2, 3)$ and the kinematical differential equations (41) are integrated numerically using the initial conditions given in the problem statement. The paths of P_1 and P_2 are identical to those shown in the upper right plot of Fig. 1, and the absolute value of $\cos\theta$ remains less than 7.64×10^{-17} throughout the simulation.

In Refs. [11] and [12] Zekovich provides examples in which velocities of two particles are to remain perpendicular to one another. However, an additional configuration constraint is imposed on P_1 and P_2; they are connected by a "fork" that allows relative translation along the line joining P_1 and P_2. In other words, P_1 is regarded as fixed in a rigid body B, and a prismatic joint makes it possible for P_2 to move on B. A relationship having the form of Eq. (36) is given, and put forth as an example of a nonlinear nonholonomic constraint equation. The development in Ref. [11] is greatly simplified by working with a set of motion variables to be defined presently; furthermore, they are used to show that the relevant nonholonomic constraint equations can be written as linear expressions.

Let perpendicular unit vectors $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$ be fixed in B such that they lie in the plane of motion of P_1 and P_2, and $\hat{\mathbf{b}}_1$ is in the direction of the prismatic joint that permits P_2 to slide on B. Unit vector $\hat{\mathbf{b}}_3$ is perpendicular to $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$, and to the plane of the motion. Four motion variables are introduced operationally by writing ${}^N\mathbf{v}^{P_1} = u_1\hat{\mathbf{b}}_1 + u_2\hat{\mathbf{b}}_2$, ${}^N\boldsymbol{\omega}^B = u_3\hat{\mathbf{b}}_3$, and ${}^B\mathbf{v}^{P_2} = u_4\hat{\mathbf{b}}_1$. The angular velocity of B in N is denoted by ${}^N\boldsymbol{\omega}^B$, and the velocity of P_2 in B is indicated by ${}^B\mathbf{v}^{P_2}$. Hence, ${}^N\mathbf{v}^{P_2} = (u_1 + u_4)\hat{\mathbf{b}}_1 + (u_2 + q_4u_3)\hat{\mathbf{b}}_2$, where q_4 is the distance between P_1 and P_2. The perpendicular velocity constraint is expressed as ${}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_1} = u_1(u_1 + u_4) + u_2(u_2 + q_4u_3) = 0$.

Zekovich begins the analysis by attaching a sharp-edged circular disk, or blade, at P_1 with the edge perpendicular to $\hat{\mathbf{b}}_1$; the resulting constraint is expressed linearly as ${}^N\mathbf{v}^{P_1} \cdot \hat{\mathbf{b}}_1 = u_1 = 0$, and the corresponding Eq. (8) in Ref. [11] is likewise linear. With $u_1 = 0$, the velocity constraint is rewritten as ${}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_1} = u_2(u_2 + q_4u_3) = 0$, which corresponds to Eq. (9) of Ref. [11]. Zekovich then notes the constraint can be satisfied in either of two ways. The first possibility is imposition of the constraint expressed by the linear equation ${}^N\mathbf{v}^{P_1} \cdot \hat{\mathbf{b}}_2 = u_2 = 0$, in which case P_1 is fixed in N and the blade at P_1 is no longer necessary. The second possibility also involves a constraint described by a linear relationship ${}^N\mathbf{v}^{P_2} \cdot \hat{\mathbf{b}}_2 = u_2 + q_4u_3 = 0$; such a restriction can be imposed by fixing a blade at P_2 with the edge orthogonal to $\hat{\mathbf{b}}_2$. The presence of perpendicular constraint forces exerted by perpendicular blades is in keeping with the result of Eqs. (33), although it contradicts the direction of $\mathbf{R}_2$ indicated in Fig. 3a of Ref. [11].

4 Other Examples

Other restrictions on the motion of two separate particles give rise to nonholonomic constraint equations that are inherently nonlinear. Constraint forces required to ensure that the velocities in N of the two particles remain parallel, or equal in magnitude, are discussed briefly. This is followed with a mention of two examples involving a single particle.

First consider the requirement that ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ be parallel to each other. The constraint can be expressed as follows. The vector $\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}$ is perpendicular to $\hat{\mathbf{n}}_3$ and to ${}^N\mathbf{v}^{P_1}$ by construction; therefore, requiring ${}^N\mathbf{v}^{P_2}$ to be parallel to ${}^N\mathbf{v}^{P_1}$ is the same as requiring

$$ {}^N\mathbf{v}^{P_2} \cdot (\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}) = 0 \tag{54} $$

This constraint equation is observed to be nonlinear in the velocity vectors because more than one velocity appears in a dot product. Differentiation with respect to t in N brings the constraint equation to the acceleration level, where it is seen to be linear in the acceleration vectors.

$$ {}^N\mathbf{a}^{P_2} \cdot (\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}) - {}^N\mathbf{a}^{P_1} \cdot (\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_2}) = 0 \tag{55} $$

In view of Eqs. (2) and (3), the constraint requires application of the forces

$$ \mathbf{C}_2 = \lambda(\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}), \qquad \mathbf{C}_1 = -\lambda(\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_2}) \tag{56} $$

to P_2 and P_1 respectively. The constraint forces $\mathbf{C}_1$ and $\mathbf{C}_2$ need not be of equal magnitudes because the constraint does not require ${}^N\mathbf{v}^{P_2}$ and ${}^N\mathbf{v}^{P_1}$ to be equal in magnitude. Moreover, $\mathbf{C}_1$ and $\mathbf{C}_2$ may have the same direction or opposite directions depending on whether the directions of ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ are opposite or the same.

The first example in Refs. [11] and [12] is similar to the preceding situation, but an additional configuration constraint is imposed on P_1 and P_2; they are connected by a rod of fixed length $2L$. It is said that the requirement of parallel velocities can be achieved in practice by attaching at the rod's midpoint a blade that is perpendicular to the rod. A relationship is given with the form of Eq. (54) written entirely in terms of scalars, and offered as an example of a nonlinear nonholonomic constraint equation. However, in this instance the constraint dictated by the blade can in fact be described by a linear nonholonomic constraint equation. There appears to be some recognition of this in Ref. [11]. The directions of the constraint forces obtained in Eqs. (56) are seen to be the same as those indicated in the diagram on the right side of Fig. 2a in Ref. [11].

Next, suppose that ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ are required to have equal magnitudes rather than parallel directions or perpendicular directions. The constraint can be expressed by the relationship

$$ {}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1} \cdot {}^N\mathbf{v}^{P_1} = 0 \tag{57} $$

which is nonlinear in the velocity vectors. The acceleration level of the constraint equation is linear in the acceleration vectors,

$$ {}^N\mathbf{a}^{P_2} \cdot {}^N\mathbf{v}^{P_2} - {}^N\mathbf{a}^{P_1} \cdot {}^N\mathbf{v}^{P_1} = 0 \tag{58} $$

According to Eqs. (2) and (3), the constraint requires application of the forces

$$\mathbf{C}_2 = \lambda\, {}^N\mathbf{v}^{P_2}, \qquad \mathbf{C}_1 = -\lambda\, {}^N\mathbf{v}^{P_1} \tag{59}$$

to P_2 and P_1 respectively. It is seen that $\mathbf{C}_1$ and $\mathbf{C}_2$ have equal magnitudes when the constraint is obeyed.

The second example in Ref. [11] involves two particles whose velocities are to remain equal in magnitude; however, an additional configuration constraint is imposed on P_1 and P_2 as they are connected by a rod of fixed length. Zekovich observes the velocities are made equal in magnitude by placing a blade at the rod's midpoint and making the edge parallel to the rod. An expression having the same form as Eq. (57), written entirely with scalars, is offered as a nonlinear nonholonomic constraint equation. As is the case with Zekovich's first example, it can easily be shown that a linear nonholonomic constraint equation describes the constraint dictated by the blade. The diagram on the right side of Fig. 2b in Ref. [11] shows a constraint force in the direction of ${}^N\mathbf{v}^{P_1}$ and the other constraint force in the direction opposite to ${}^N\mathbf{v}^{P_2}$; this result can be made to agree with Eqs. (59) by renaming the two particles.

Jankowski has developed an approach for dealing with constraint equations that are not necessarily linear in acceleration. A procedure is set forth in Ref. [17] for forming dynamical equations of motion in which Lagrange multipliers do appear, and then the multipliers are eliminated by employing an orthogonal complement matrix to obtain a reduced set of equations. As mentioned previously, the paper concludes with an example involving a single particle P. It is readily demonstrated that Eqs. (4) and (5) can be used to obtain the results reported in Ref. [17] when the magnitude of the velocity ${}^N\mathbf{v}^P$ of P in N must have a prescribed time history; that is, ${}^N\mathbf{v}^P \cdot {}^N\mathbf{v}^P - v(t)^2 = 0$. Moreover, inspection of this constraint equation at the acceleration level indicates the constraint force applied to P is in the direction of ${}^N\mathbf{v}^P$, and Jankowski reaches the same conclusion. However, Eqs. (4) and (5) are not applicable to the subsequent case in which the magnitude of the acceleration ${}^N\mathbf{a}^P$ of P in N is a prescribed function of the time t, ${}^N\mathbf{a}^P \cdot {}^N\mathbf{a}^P - a(t)^2 = 0$

5 Appell's Particle

As mentioned earlier, the literature contains ample discussion of an example proposed by Appell in which a single particle must move in a uniform gravitational field so as to satisfy a nonlinear nonholonomic constraint equation. In connection with this example, a final brief demonstration of the use of Eqs. (4) and (5) shows that they lead to results obtained by Smith[1] and Van Dooren (Ref. [19]).

Three motion variables u_1, u_2, and u_3 are introduced such that the velocity ${}^N\mathbf{v}^P$ in a Newtonian reference frame N of a particle P is written as

$$ {}^N\mathbf{v}^P = u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2 + u_3\hat{\mathbf{n}}_3 \tag{60} $$

where $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, and $\hat{\mathbf{n}}_3$ are a right-handed set of mutually perpendicular unit vectors fixed in N. Appell's restriction on the velocity of P is often expressed by the relationship

$$ {u_3}^2 = a^2({u_1}^2 + {u_2}^2) \tag{61} $$

where a is a constant. It is pointed out by Smith that the relationship describes a requirement for the angle γ between ${}^N\mathbf{v}^P$ and $\hat{\mathbf{n}}_3$, the vertical direction, to remain constant. In fact, the constant a is $\cos\gamma/\sin\gamma$. The nonlinear nonholonomic constraint equation is differentiated with respect to time to bring it to the acceleration level

$$ 2u_3\dot{u}_3 = 2a^2(u_1\dot{u}_1 + u_2\dot{u}_2) \tag{62} $$

where it is linear in $\dot{u}_1$, $\dot{u}_2$, and $\dot{u}_3$; it can be rewritten as

$$ {}^N\mathbf{a}^P \cdot \hat{\mathbf{n}}_3 - \frac{a^2}{u_3}(u_1\, {}^N\mathbf{a}^P \cdot \hat{\mathbf{n}}_1 + u_2\, {}^N\mathbf{a}^P \cdot \hat{\mathbf{n}}_2) = {}^N\mathbf{a}^P \cdot \left[\hat{\mathbf{n}}_3 - \frac{a^2}{u_3}(u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2)\right] = 0 \tag{63} $$

where ${}^N\mathbf{a}^P$ is the acceleration of P in N. Inspection of this equation according to Eqs. (2) and (3) indicates that a constraint force $\mathbf{C}$ must be applied to P such that the force is parallel to the vector within the square brackets; that is,

$$ \mathbf{C} = \lambda\left[\hat{\mathbf{n}}_3 - \frac{a^2}{u_3}(u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2)\right] \tag{64} $$

[1] Smith, C., "Comments on Geometric Constraints, Virtual Displacements, and Ideal Constraint Forces," private communication, Sept. 25, 2002.

This result is in agreement with what is presented by Smith, who shows that $\mathbf{C} \cdot {}^N\mathbf{v}^P = 0$ when ${}^N\mathbf{v}^P$ obeys the constraint.

The gravitational force acting on P is denoted by $\mathbf{f} = -mg\hat{\mathbf{n}}_3$ where m is the mass of P and the constant g represents the gravitational force per unit mass. Three dynamical equations of motion obtained with Eqs. (4) can be written in terms of vectors as $\hat{\mathbf{n}}_r \cdot (\mathbf{f} + \mathbf{C} - m\,{}^N\mathbf{a}^P) = 0$ $(r = 1, 2, 3)$, or in terms of scalars

$$m\dot{u}_1 = -\lambda a^2 u_1/u_3, \qquad m\dot{u}_2 = -\lambda a^2 u_2/u_3, \qquad m\dot{u}_3 = \lambda - mg \tag{65}$$

in which case they resemble certain expressions found by Smith. When one substitutes u_3 obtained from the constraint equation (61), the results are identical to Eqs. (3.7) of Ref. [19],

$$m\dot{u}_1 = -\lambda a \frac{u_1}{\sqrt{{u_1}^2 + {u_2}^2}}, \qquad m\dot{u}_2 = -\lambda a \frac{u_2}{\sqrt{{u_1}^2 + {u_2}^2}}, \qquad m\dot{u}_3 = \lambda - mg \tag{66}$$

The fourth relationship needed to determine the unknowns $\dot{u}_1$, $\dot{u}_2$, $\dot{u}_3$, and λ is provided by Eq. (62); when it is solved for $\dot{u}_3$ and substitution is performed in the third of Eqs. (66), one obtains

$$\lambda = mg + m\frac{a^2}{u_3}(u_1\dot{u}_1 + u_2\dot{u}_2) = mg - \frac{a}{\sqrt{{u_1}^2 + {u_2}^2}}\left[\frac{\lambda a({u_1}^2 + {u_2}^2)}{\sqrt{{u_1}^2 + {u_2}^2}}\right] = mg - \lambda a^2 \tag{67}$$

where the second step is made with the aid of Eq. (61) together with the first and second of Eqs. (66). A solution for λ is now at hand, and it can be used as a replacement in the first and second of Eqs. (66) to yield

$$\lambda = \frac{mg}{1 + a^2} = mg\sin^2\gamma \tag{68}$$

$$\dot{u}_1 = -\frac{gau_1}{(1 + a^2)\sqrt{{u_1}^2 + {u_2}^2}} = -\frac{g\sin\gamma\cos\gamma\, u_1}{\sqrt{{u_1}^2 + {u_2}^2}} \tag{69}$$

$$\dot{u}_2 = -\frac{gau_2}{(1 + a^2)\sqrt{{u_1}^2 + {u_2}^2}} = -\frac{g\sin\gamma\cos\gamma\, u_2}{\sqrt{{u_1}^2 + {u_2}^2}} \tag{70}$$

The dynamical equations of motion (69) and (70) from which λ has been eliminated can be obtained directly, instead, by resorting to Eqs. (5). After embedding the acceleration level constraint equation in ${}^N\mathbf{a}^P$,

$${}^N\mathbf{a}^P = \dot{u}_1\hat{\mathbf{n}}_1 + \dot{u}_2\hat{\mathbf{n}}_2 + \frac{a(u_1\dot{u}_1 + u_2\dot{u}_2)}{\sqrt{{u_1}^2 + {u_2}^2}}\hat{\mathbf{n}}_3 \tag{71}$$

the required nonholonomic partial accelerations of P in N are readily identified to be

$$
{}^N\tilde{\mathbf{a}}_1^P = \hat{\mathbf{n}}_1 + \frac{au_1}{\sqrt{{u_1}^2 + {u_2}^2}}\hat{\mathbf{n}}_3, \qquad {}^N\tilde{\mathbf{a}}_2^P = \hat{\mathbf{n}}_2 + \frac{au_2}{\sqrt{{u_1}^2 + {u_2}^2}}\hat{\mathbf{n}}_3 \tag{72}
$$

The two equations of interest are then produced from ${}^N\tilde{\mathbf{a}}_r^P \cdot (\mathbf{f} + \mathbf{C} - m\,{}^N\mathbf{a}^P) = {}^N\tilde{\mathbf{a}}_r^P \cdot (\mathbf{f} - m\,{}^N\mathbf{a}^P) = 0$ $(r = 1, 2)$. Although some effort is required because the equations are coupled in $\dot{u}_1$ and $\dot{u}_2$, Eqs. (69) and (70) are recovered.

6 A System Containing a Rigid Body

There are certain concepts that the exposition in Sec. 2 has in common with that of Ref. [10]. The authors of that work recognize constraint equations that are nonlinear at the velocity level become linear at the acceleration level, and they note the relationship between partial acceleration and partial velocity expressed in Eqs. (9). They make use of these observations to form equations of motion that are equivalent to Eqs. (4), and form generalized constraint forces that are expressed with the final term in Eqs. (23). It is pointed out that the unknown multipliers representing the constraint forces can be eliminated and a reduced set of equations of motion can be obtained. There are, however, a number of differences between what is presented here and in Ref. [10]. In that work, the development is restricted to the case where each motion variable is defined as the time derivative of a generalized coordinate, and remainder terms such as ${}^N\mathbf{v}_t^{P_i}$ or ${}^N\tilde{\mathbf{v}}_t^{P_i}$ are not accounted for. Constraint forces are not constructed from vector forms of the constraint equations as they are here in connection with Eqs. (2) and (3); therefore an explicit relationship between the multipliers and constraint forces is not provided. Their development requires partial velocities to be expressed in a vector basis fixed in an inertial reference frame, which is not necessarily convenient or efficient. The most significant difference is that, although their reduced equations of motion are similar to Eqs. (5), reduction is accomplished by premultiplication with a nonunique orthogonal complement matrix that can be formed in a variety of ways; in contrast, the nonholonomic partial accelerations proposed here are unique once motion variables have been chosen, and they are formed by the same definite process of inspection used to obtain partial velocities. Finally, the Appell-Hamel mechanism is used to illustrate their method, even though it is known that the nonholonomic constraint equations can be expressed in a linear form.

The apparatus of Ref. [10] deals with rigid bodies rather than sets of individual particles; the development of the present approach is completed by fashioning rigid-body theorems for the foregoing results.

When particles $P_1, \ldots, P_\beta$ make up a rigid body B, the acceleration ${}^N\mathbf{a}^{P_i}$ in N of a generic particle P_i of B can be written in terms of the angular acceleration ${}^N\boldsymbol{\alpha}^B$ of B in N, the angular velocity ${}^N\boldsymbol{\omega}^B$ of B in N, and the acceleration ${}^N\mathbf{a}^{B^\star}$ in N of $B^\star$, the mass center of B,

$$ {}^N\mathbf{a}^{P_i} = {}^N\mathbf{a}^{B^\star} + {}^N\boldsymbol{\alpha}^B \times \mathbf{r}_i + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i) \qquad (i = 1, \ldots, \beta) \tag{73} $$

where $\mathbf{r}_i$ is the position vector from $B^\star$ to P_i. Now, ${}^N\boldsymbol{\alpha}^B$ can be expressed uniquely as

$$ {}^N\boldsymbol{\alpha}^B = \sum_{r=1}^{c} {}^N\tilde{\boldsymbol{\alpha}}_r^B\, \dot{u}_r + {}^N\tilde{\boldsymbol{\alpha}}_t^B \tag{74} $$

where ${}^N\tilde{\boldsymbol{\alpha}}_r^B$ is called the rth *nonholonomic partial angular acceleration* of B in N. Substitution from this relationship and from Eqs. (8) into (73) yields

$$ \begin{aligned} \sum_{r=1}^{c} {}^N\tilde{\mathbf{a}}_r^{P_i}\, \dot{u}_r + {}^N\tilde{\mathbf{a}}_t^{P_i} &= \sum_{r=1}^{c} {}^N\tilde{\mathbf{a}}_r^{B^\star}\, \dot{u}_r + {}^N\tilde{\mathbf{a}}_t^{B^\star} \\ &\quad + \left(\sum_{r=1}^{c} {}^N\tilde{\boldsymbol{\alpha}}_r^B\, \dot{u}_r + {}^N\tilde{\boldsymbol{\alpha}}_t^B \right) \times \mathbf{r}_i + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i) \quad (i = 1, \ldots, \beta) \end{aligned} \tag{75} $$

from which one obtains

$$ {}^N\tilde{\mathbf{a}}_t^{P_i} = {}^N\tilde{\mathbf{a}}_t^{B^\star} + {}^N\tilde{\boldsymbol{\alpha}}_t^B \times \mathbf{r}_i + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i) \qquad (i = 1, \ldots, \beta) \tag{76} $$

and

$$ {}^N\tilde{\mathbf{a}}_r^{P_i} = {}^N\tilde{\mathbf{a}}_r^{B^\star} + {}^N\tilde{\boldsymbol{\alpha}}_r^B \times \mathbf{r}_i \qquad (r = 1, \ldots, c;\ i = 1, \ldots, \beta) \tag{77} $$

The latter relationship is the nonholonomic partial acceleration analog to nonholonomic partial velocity expressions like Eqs. (4.6.5) and (4.11.16) in Ref. [22] used in the case of simple nonholonomic systems to obtain contributions of B to $\tilde{F}_r$ and $\tilde{F}_r^\star$. Hence, the contribution of B to $\tilde{\tilde{F}}_r$ is given by

$$
\begin{aligned}
(\tilde{\tilde{F}}_r)_B &\triangleq \sum_{i=1}^{\beta} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \mathbf{R}_i \\
&= \sum_{i=1}^{\beta} \left({}^N\tilde{\mathbf{a}}_r^{B^\star} + {}^N\tilde{\boldsymbol{\alpha}}_r^{B} \times \mathbf{r}_i \right) \cdot \mathbf{R}_i = {}^N\tilde{\mathbf{a}}_r^{B^\star} \cdot \sum_{i=1}^{\beta} \mathbf{R}_i + {}^N\tilde{\boldsymbol{\alpha}}_r^{B} \cdot \sum_{i=1}^{\beta} \mathbf{r}_i \times \mathbf{R}_i \\
&= {}^N\tilde{\mathbf{a}}_r^{B^\star} \cdot \mathbf{R} + {}^N\tilde{\boldsymbol{\alpha}}_r^{B} \cdot \mathbf{T} \qquad (r = 1, \ldots, c)
\end{aligned} \tag{78}
$$

where the set of all contact and distance forces $\mathbf{R}_i$ acting on the particles of B is equivalent to a force $\mathbf{R}$ whose line of action passes through $B^\star$, together with a couple whose torque is $\mathbf{T}$. The constraint forces and torques that must be applied to B in order to satisfy nonlinear nonholonomic constraint equations may be included in $\mathbf{R}$ and $\mathbf{T}$, or they may be omitted; in either case they will not contribute to $(\tilde{\tilde{F}}_r)_B$. With a similar exercise the contribution of B to $\tilde{\tilde{F}}_r^\star$ is found to be

$$
\begin{aligned}
(\tilde{\tilde{F}}_r^\star)_B &\triangleq -\sum_{i=1}^{\beta} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot m_i \, {}^N\mathbf{a}^{P_i} \\
&= -\sum_{i=1}^{\beta} \left({}^N\tilde{\mathbf{a}}_r^{B^\star} + {}^N\tilde{\boldsymbol{\alpha}}_r^{B} \times \mathbf{r}_i \right) \cdot m_i \, {}^N\mathbf{a}^{P_i} \\
&= -{}^N\tilde{\mathbf{a}}_r^{B^\star} \cdot \sum_{i=1}^{\beta} m_i \, {}^N\mathbf{a}^{P_i} - {}^N\tilde{\boldsymbol{\alpha}}_r^{B} \cdot \sum_{i=1}^{\beta} \mathbf{r}_i \times m_i \, {}^N\mathbf{a}^{P_i} \\
&= {}^N\tilde{\mathbf{a}}_r^{B^\star} \cdot \mathbf{R}^\star + {}^N\tilde{\boldsymbol{\alpha}}_r^{B} \cdot \mathbf{T}^\star \qquad (r = 1, \ldots, c)
\end{aligned} \tag{79}
$$

where $\mathbf{R}^\star$ and $\mathbf{T}^\star$ are, respectively, the well known inertia force and inertia torque for B in N formed for use with Kane's method.

The procedure for obtaining a minimal set of dynamical equations of motion for a complex nonholonomic system is seen to bear a very close resemblance to Kane's method for simple nonholonomic systems, the only difference being that one uses ${}^N\tilde{\mathbf{a}}_r^{B^\star}$ and ${}^N\tilde{\boldsymbol{\alpha}}_r^{B}$ $(r = 1, \ldots, c)$ rather than ${}^N\tilde{\mathbf{v}}_r^{B^\star}$ and ${}^N\tilde{\boldsymbol{\omega}}_r^{B}$ $(r = 1, \ldots, p)$.

One may be interested in the constraint forces acting on a rigid body, and therefore form equations of motion according to Eqs. (4). In that event it becomes desirable to adapt the process of inspecting a constraint equation written at the acceleration level so that one may identify the direction of a constraint force and the point to which it is applied, together with the direction of a constraint torque and the body upon which it is exerted.

In a constraint equation having the form of (2), the terms associated with $P_1, \ldots, P_\beta$ can be rewritten:

$$\begin{aligned}
&\sum_{i=1}^{\beta} {}^N\mathbf{a}^{P_i} \cdot \mathbf{W}_{is} + Z_s \\
&\quad = \sum_{i=1}^{\beta} [{}^N\mathbf{a}^Q + {}^N\boldsymbol{\alpha}^B \times \mathbf{r}_i + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i)] \cdot \mathbf{W}_{is} + Z_s \\
&\quad = {}^N\mathbf{a}^Q \cdot \sum_{i=1}^{\beta} \mathbf{W}_{is} + {}^N\boldsymbol{\alpha}^B \cdot \sum_{i=1}^{\beta} \mathbf{r}_i \times \mathbf{W}_{is} + \sum_{i=1}^{\beta} [{}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i)] \cdot \mathbf{W}_{is} + Z_s \\
&\quad \triangleq {}^N\mathbf{a}^Q \cdot \mathbf{W}_s + {}^N\boldsymbol{\alpha}^B \cdot \boldsymbol{\tau}_s + Z'_s \qquad (s = 1, \ldots, l)
\end{aligned} \tag{80}$$

where $\mathbf{r}_i$ is the position vector from a point Q fixed in B to P_i $(i = 1, \ldots, \beta)$. As discussed in connection with Eqs. (2) and (3), the appearance of the vector $\mathbf{W}_{is}$ in Eqs. (80) requires the application of a constraint force $\mathbf{C}_{is} = \lambda_s \mathbf{W}_{is}$ to P_i. After selecting the line of action of $\mathbf{W}_{is}$ such that it passes through P_i, and defining the resultants

$$\mathbf{W}_s \triangleq \sum_{i=1}^{\beta} \mathbf{W}_{is}, \qquad \mathbf{C}_s \triangleq \sum_{i=1}^{\beta} \mathbf{C}_{is} \qquad (s = 1, \ldots, \ell) \tag{81}$$

the set of forces $\mathbf{C}_{1s}, \ldots, \mathbf{C}_{\beta s}$ applied to B is regarded as equivalent to a single force $\mathbf{C}_s$ whose line of action passes through Q, together with a couple whose torque is $\mathbf{T}_s$. The resultant $\mathbf{C}_s$ is given by

$$\mathbf{C}_s = \sum_{i=1}^{\beta} \mathbf{C}_{is} = \sum_{i=1}^{\beta} \lambda_s \mathbf{W}_{is} = \lambda_s \mathbf{W}_s \qquad (s = 1, \ldots, \ell) \tag{82}$$

and the torque $\mathbf{T}_s$ is equal to the moment of $\mathbf{C}_{1s}, \ldots, \mathbf{C}_{\beta s}$ about Q,

$$\mathbf{T}_s = \sum_{i=1}^{\beta} \mathbf{r}_i \times \mathbf{C}_{is} = \sum_{i=1}^{\beta} \mathbf{r}_i \times \lambda_s \mathbf{W}_{is} = \lambda_s \boldsymbol{\tau}_s \qquad (s = 1, \ldots, \ell) \tag{83}$$

where $\boldsymbol{\tau}_s$ is the moment of $\mathbf{W}_{1s}, \ldots, \mathbf{W}_{\beta s}$ about Q,

$$\boldsymbol{\tau}_s \triangleq \sum_{i=1}^{\beta} \mathbf{r}_i \times \mathbf{W}_{is} \qquad (s = 1, \ldots, \ell) \tag{84}$$

One can therefore inspect a constraint equation written at the acceleration level and conclude that the appearance of the dot product ${}^N\mathbf{a}^Q \cdot \mathbf{W}_s$ requires that B is subject to a constraint force $\mathbf{C}_s = \lambda_s \mathbf{W}_s$ applied to Q, and the appearance of the dot product ${}^N\boldsymbol{\alpha}^B \cdot \boldsymbol{\tau}_s$ means B must be acted upon by a couple whose constraint torque is $\mathbf{T}_s = \lambda_s \boldsymbol{\tau}_s$ $(s = 1, \ldots, \ell)$.

7 Conclusions

Motion constraint equations that are nonlinear in velocity, and thus represent nonclassical servo-constraints or program constraints, become linear in acceleration after they are differentiated with respect to time in N. Hence, one can use the following method to identify a set of constraint forces necessary to ensure that a mechanical system composed of particles and rigid bodies obeys the restrictions. The appearance of a dot product such as ${}^N\mathbf{a}^P \cdot \mathbf{W}$, where ${}^N\mathbf{a}^P$ is the acceleration of a point P in a Newtonian reference frame N, indicates a constraint force $\lambda\mathbf{W}$ must be applied to P; λ is a scalar multiplier. When the mechanical system contains a rigid body B, the angular acceleration ${}^N\boldsymbol{\alpha}^B$ of B in N may appear in a dot product such as ${}^N\boldsymbol{\alpha}^B \cdot \boldsymbol{\tau}$ in a constraint equation written at the acceleration level, in which case a constraint couple whose torque is $\lambda\boldsymbol{\tau}$ must be exerted on B.

This paper furnishes additional evidence that the method is comprehensive; as shown in Ref. [21], it is useful also in connection with configuration constraints described by holonomic constraint equations, and motion constraints expressed by nonholonomic constraint equations that are linear in velocity. As demonstrated here by several examples, this method is especially advantageous in cases where the required direction of a constraint force is not otherwise obvious.

The use of partial accelerations, instead of the partial velocities normally employed with Kane's method, leads to the development of two new approaches for deriving equations of motion for a complex nonholonomic system; that is, a system subject to constraints expressed at the velocity level with equations that are nonlinear in velocity. The two algorithms enable construction of dynamical equations that either do or do not contain evidence of the constraint forces, according to the interests of the analyst.

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